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The components of a quantum computer are quantum subsystems which have a complex internal structure. This structure is determined by short-range interactions which are appropriately described in terms of local gauge fields of the first kind. Any modification of their state would produce, in general, a new type of internal error, called local error, in the quantum state of the computer. We suggest that the general treatement of the local errors produced by a gauge multiplet can be done in the framework of the Algebraic Quantum Field Theory. A recovery operator is constructed from the first principles.

Quantum computers are operational providing the errors in stocking and manipulating the information are known and can be controled. Classifying and correcting the quantum errors is crucial for practical reasons as well as for defining rigorously an Universal Quantum Computer. Most of the errors that alter the memory register or the output of the quantum computation are due to the small scale structure of the computer subsystems and to their sensitivity to the interaction with the environment and with each other. The decoherence of the quantum state and the uncertainty in the unitary evolution have been studied for spin $\frac{1}{2}$-systems [1–11] and higher spin systems [12–15]. Other effects that can alter the information as the quantum chaos [16–18], the self-interaction among various qubits [17,19] and the changes in the continuous variables of the system [20–23] have been considered and the codes to correct them have been written down. A general observation that can be drawn from the above studies is that long-range (electromagnetic) and/or non-linear interactions are responsible for the alteration of the quantum state of the processing unit. This statement resumes the idea that the quantum computers are "small, sensitive and easily perturbed". However, one may ask whether the short-range interactions affect in any way the precision of quantum computations. At Quantum Mechanics scale (10^{-15}m), the external errors and the ones provocated by the preparation of the initial state, the manipulation of the computer or the reading of the final state are discussed by assuming that the "computer parts" (atoms, molecules or nuclei) are stable systems. Nevertheless, these components are, at their turn, quantum objects with an internal structure equally determined by short-range interactions. Even if, in most of the practical situations, their life-time is large enough, they can have their internal state changed (like in spontaneous emission or decayment) . The modification of the internal state of any subsystem of a computer may, in general, alter the state of the whole system inducing a new type of quantum errors. The purpose of this letter is to put this question into a more formal frame and to discuss the possibility of constructing recovery operators for this type of errors.

The parts of a quantum computer are complex subsystems hold together by internal interactions. If we focus to nuclei, which form subsystems of any quantum computer considered up to now, the most important role is played by gauge fields which produce local interactions and are nonlinear. In general, a nucleus can be described as a state of the gauge field. Consequently, one can define the state of the computer as a (more complicate) state in the Hilbert space of the field theory. A change of the internal state of a nucleus as, for example, a spontaneous emission that would carry away spin quantum number, may induce a modification of the quantum state of the computer. This transition can occur under the action of an external system, as has been considered up to now, or of gauge fields. We are going to focus our attention on the later case. In the space representation, the transition of a nucleus from an internal state to another is a local phenomenon due to local (short-range) fields. Therefore, it is natural to assume that: i) there are local errors produced by local field operators in a local quantum gauge field theory and ii) the state of the computer is described by a vector in the Hilbert space of the field theory. The physics of the field theory is contained in the observables which are constructed from field operators.

There are three basic principles that such of theory should obey : *the principle of locality*, *the principle of causality* and *the principle of gauge invariance*. They guarantee that the errors in the quantum computer are local concearning a single subsystem of the computer and that the quantum field theory we are dealing with is a known one. Formally, the principle of locality states that to any open and finite extended region of space-time Σ

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one can associate a *-algebra of observables $\mathcal{O}(\Sigma)$ and a *-algebra of fields $\mathcal{F}(\Sigma)$. (The involution * is necessary in order to define the conjugates of fields. In what follows we will denote it by † as is usual.) One can define global algebras over any finite volume of the computer or any of its subsystems by considering the norm closed unions of the open neighbourhoods covering that region

$$\mathcal{O} = \cup\mathcal{O}(\Sigma) \quad , \quad \mathcal{F} = \cup\mathcal{F}(\Sigma). \tag{1}$$

The principle of causality states that for any two space-like separated regions of space-time Σ_1 and Σ_2 the corresponding algebras of observables commute with each other, i.e. $[\mathcal{O}(\Sigma_1), \mathcal{O}(\Sigma_2)] = 0$ (the observables must be bosonos, but the fields may be fermions.) The principle of gauge invariance tells us that there is a representation of a gauge group $\mathcal{G}$ (denoted for simplicity with the same letter) that acts on the algebra of fields $\mathcal{F}(\Sigma)$. The algebra $\mathcal{O}(\Sigma)$ is gauge invariant. Since we assume that the observables are constructed out of fields, $\mathcal{O}(\Sigma)$ is the gauge invariant part of the algebra $\mathcal{F}(\Sigma)$. If we want to describe the effects of short-range interactions we can limit ourselves to the gauge groups $SU(2)$ or $SU(3)$. However, we are going to keep the discussion more general by working with arbitrary compact gauge groups. In what follows $\mathcal{G}$ denote a compact gauge group that defines a gauge symmetry of the first kind, i. e. which excludes the long range interactions. Under these assumptions, the setting of the problem is the Algebraic Quantum Field Theory [24]. As was shown in [25–28] the algebra $\mathcal{F}$ is generated by a gauge multiplet $\{\Psi_i\}$, $i = 1, 2, \ldots, n$ and $\mathcal{O}$. From the principle of locality and the principle of causality it follows that the multiplet should generate a Cuntz algebra O_d [25,26], i.e. it satisfies the following relations [25,26]

$$\sum_i \Psi_i \Psi_i^\dagger = 1$$
$$\Psi_i^\dagger \Psi_j = \delta_{ij}. \tag{2}$$

There is a canonical endomorphism of the algebra $\mathcal{O}$ which defines the density matrix for any field $A \in \mathcal{F}$

$$\rho(A) = \sum_i \Psi_i A \Psi_i^\dagger \tag{3}$$

and which satisfies the following relation for any Ψ and any A from $\mathcal{F}$

$$\Psi A = \rho(A)\Psi. \tag{4}$$

It has been shown in [25,26,29] that the mathematical structure of $\mathcal{F}$ is that of a cross-product algebra of O_d by the action of $\mathcal{G}$.

Now let us assume that the computer is prepared initially in the pure state $|\phi_I\rangle$ which is a vector in the Hilbert space of the field theory and is determined by the states of the computers subsystems (e.g, nuclei). The local errors which we are considering here represent a new state obtained by acting on the initial state with products of the field operators, i. e. elements of the algebra $\mathcal{F}$. For simplicity, we will consider in what follows only the action of the generators $\{\Psi_i\}$ of O_d. Thus, the system evolves to an error state ρ_F given by

$$\rho_F = \sum_i \Psi_i \rho_I \Psi_i^\dagger, \tag{5}$$

where ρ_I is the density matrix associated to the initial vector state. We remark from (2) that the set of operators $\{\Psi_i\}$ form a superoperator [30]. Due to this structure, it is appropriate to correct the error by using the recovery operator method, i.e. to construct a set of operators $\{R_a\}$ that projects the wrong state ρ_F back to the initial state [30]. To this end, note that one can associate a projector P_i to each field operator Ψ_i defined by the following relation

$$P_i = \Psi_i \Psi_i^\dagger \quad , \quad P_i^2 = P_i^\dagger = P_i, \tag{6}$$

with the following action on any density matrix

$$\rho \;\mapsto\; \rho_P = \sum_i P_i \rho P_i. \tag{7}$$

The relations (2) imply that P_i's project on to orthogonal directions in the Hilbert space

$$P_i P_j = \delta_{ij} P_j. \tag{8}$$

The full set $\{P_i\}$ projects the state ρ_F to itself and map any other state $\rho \neq \rho_F$ to a different state. Therefore, they are not suited to recover the initial state. However, since the states obtained by projecting the final state through each P_i are orthogonal on each other, one could seek a linear combination of them that is the initial state of the computer. Equivalently, one can look for a particular linear combinations of the projectors that projects the final state on to the inital one. This combination defines the recovery operator. The projected state is equal to the initial state if its norm is equal to the norm of the initial state. Let us pursue these ideas by requiring that the fidelity of the state obtained by acting with the recovery operator on ρ_F be maximum in the direction of $|\phi_I\rangle$, i.e. equal to the norm of $|\phi_I\rangle$. The fidelity of a state ρ is defined as the square of the norm of the element matrix of it in the inital state

$$F(\phi_I, \rho) = \langle \phi_I \, | \rho | \, \phi_I \rangle \, . \tag{9}$$

Consider the following linear combinations of projectors

$$R_a = \sum_i \alpha_{ai} P_i, \tag{10}$$

where $a = 1, 2, \ldots, M$ belongs to a discrete and finite set and α_{ai} are complex numbers. $\{R_a\}$ map ρ_F to the

state ρ_R by the usual action of the operators on matrix densities

$$\rho_F \;\mapsto\; \rho_R = \sum_a R_a \rho_F R_a^\dagger. \tag{11}$$

The initial state is recovered by R_a's if

$$F(\phi_I, \rho_R) = \langle \phi_I \left| \rho_I \right| \phi_I \rangle . \tag{12}$$

Assume for simplicity that the norm of the initial state is one. Then (12) is equivalent to the following relation

$$\sum_a \sum_i \left| \alpha_{ai} \right|^2 \left| \Psi_{iI} \right|^2 = 1, \tag{13}$$

where $\Psi_{iI} = \langle \phi_I \left| \Psi_i \right| \phi_I \rangle$. The relation (13) represents a constraint on the moduli of the complex coefficients α_{ai} in terms of the known gauge multiplet and the initial state. If a whole subset C of the Hilbert space is recovered by the operators R_a's, then (13) should hold for any $|\phi_I\rangle$ from C. Suppose that C form a linear subspace of finite dimension k of the Hilbert space of the field theory. By picking up a basis $\{|e_A\rangle\}$ of it, one can determine the constants α_{ai}'s, and consequently the operators R_a's, up to some phase factors. This can be achieved by requiring that any of the elements of the basis be recovered by R_a's and solving the corresponding linear system

$$\sum_a \sum_i \left| \alpha_{ai} \right|^2 \left| \Psi_{iA} \right|^2 = 1 \quad , \quad A = 1, 2, \ldots, k. \tag{14}$$

The number M of the operators R_a depends on the dimension k of C and one should seek for its minimum value $M = k$ for which there are sufficient equations in (14) to determine the moduli of the coefficients α_{ai}.

The relations (14) defines a linear-antilinear form on the vectors of the subspace C. We would like to know what the transformation of the basis $\{|\epsilon_A\rangle\}$ that preserve (14) are. Consider an arbitrary linear transformation

$$|f_B\rangle = \sum_C \theta_{BC} \left| e_A \right\rangle , \tag{15}$$

where $B, C = 1, 2, \ldots, M$ and θ_{BC} are the complex elements of the transformation matrix between the two basis. If the relations (14) are to hold for any of the elements of the new basis on can find by an elementary algebra the following relation

$$\sum_C \left| \theta_{BC} \right|^2 + \sum_{C \neq D} \theta^*_{BC} \theta_{BD} \left| \Psi_{iCD} \right| = 1, \; \forall B = 1, 2, \ldots M, \tag{16}$$

where $|\Psi_{iCD}| = \langle f_C \left| \Psi_i \right| f_D \rangle$. The equation (16) defines the relations among the elements θ_{BC} of the transformation matrix and imposes constraints on the sign of the biproducts $\theta^*_{BC}\theta_{CD}$. Indeed, since $|\Psi_{iCD}|$ represents the probability of transition from the state $|f_D\rangle$ to the state $|f_C\rangle$ under the action of the field operator Ψ_i, it is a positive real number or zero. Consequently, if $\sum_C \left| \theta_{BC} \right|^2 = 1$ for any $B = 1, 2, \ldots, M$, then the signs of the products $\theta^*_{BC}\theta_{CD}$ must alternate or $|\Psi_{iCD}|$ must be all zero. Similar anaysis can be done if the first term in (16) is smaller or greater than one. The transformations that satisfy (16) form a group. The recovery operator is invariant under these transformations, i. e. the coefficients α_{ai} do not depend on the vectors of the code space C, but only on the gauge multiplet.

There is one more constraint that can be imposed naturally on the recovery operator and it comes from the gauge structure of the theory. Firstly, note that the fidelity is gauge invariant

$$F(\phi'_I, \rho'_R) = F(\phi_I, \rho_R), \tag{17}$$

where the transformations of the states and fields is given by the following relations

$$|\psi\rangle \;\mapsto |\psi'\rangle = g \left| \phi_I \right\rangle \tag{18}$$

$$\Psi_i \mapsto \Psi'_i = g \Psi_i g^{-1} \tag{19}$$

for any $g \in \mathcal{G}$. Since the norm of the initial state is gauge invariant, the equations (13) and (14) are gauge invariant, too, as expected. Consider next the following isometry of the algebra $\mathcal{F}$ [29]

$$S = \frac{1}{\sqrt{d!}} \sum_{q \in P(d)} \mathrm{sign}(q) \Psi_{q(1)} \Psi_{q(2)} \ldots \Psi_{q(d)}, \tag{20}$$

where $P(d)$ is the permutation group of d elements. S is a gauge invariant object of $\mathcal{F}$ [28,29] and it acts on the initial state as

$$\rho_I \mapsto \rho_S = S \rho_I S^\dagger. \tag{21}$$

Now consider the following transformations: act firstly with a gauge transformation (18) and (19) and then apply the gauge invariant isometry (20). Since S is an gauge invariant operator, we require that the recovery operator R'_a obtained from (19) maximizes the fidelity on the gauge transformed initial state $|\phi'_I\rangle$

$$F(\phi'_I, \rho'_{RS}) = \langle \phi'_I \left| \rho'_S \right| \phi'_I \rangle . \tag{22}$$

It is an elementary exercise to show that (22) is equivalent to the same equation for the untrasformed objects. After a simple algebra we obtain the following relation from (22)

$$\sum_{a=1}^{M} \sum_{q,r \in P(d)} \frac{1}{d!} \mathrm{sign}(q) \mathrm{sign}(r) \alpha_{aq(1)} \alpha^*_{ar(1)} \times \left\langle \Psi_{q(1)} \cdots \Psi_{q(d)} \right\rangle_I \left\langle \Psi^\dagger_{r(d)} \cdots \Psi^\dagger_{r(1)} \right\rangle_I = 1, \tag{23}$$

where $\langle \cdots \rangle_I$ is a shorthand notation for $\langle \phi_I | \cdots | \phi_I \rangle$. We interpret (23) as a constraint on the complex coefficients α_{aI}.

The equations (13), (14) and (23) represent the main result of the present work. They are based on the ansatz (10). While (13) and (14) provide the linear equations for determining the coefficents in the recovery operator $\mathcal{R} = \{R_a\}$ for $a = 1, 2, \ldots, M = k$ up to some phase factors, the relation (23) is obtained from the gauge symmetry of the field operators. It states that under an arbitrary gauge transformation and a gauge invariant transformation the recovery operator remains so. Since the phases of the complex coefficients α_{ai} are not constrained by any relation, we may fix them to zero to get real coefficients. The dimension of the code space C fixes the number of operators R_a. The equations (13), (14) and (23) were determined from general principles of the Algebraic Quantum Field Theory and from the requirement that the fidelity is maximized for any state from the code space. The relation (16) defines the group of transformations that preserve the recovery operator $\mathcal{R}$ in the sense that it is determined only by the gauge multiplet and not by the initial states of the system. The interpretation of $\mathcal{R}$ is that of the recovery operator for the internal errors produced by a local gauge multiplet. They alter the state of the computer either by changing the states of quantum subsystems or by changing locally the general state of memory register. The physics behind the present description is that of modifications in the internal state of the subsystems of the computer induced by short-distance interactions. Mathematically, we have been working in the frame of the Algebraic Quantum Field Theory.

The discussion about the errors induced by the algebra of observables $\mathcal{O}$ stays open. On general grounds, we would expect that they produce similar errors as the local fields. However, the algebra of observables does not have, in general, the structure of the Cuntz algebra. Another interesting class of internal error is that produced by non-linearities in the gauge fields that are responsible for the internal structure of the quantum components of the computer. We hope to clarify these issues somewhere else [35].

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